DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

300. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Stillwater, Okla.

If α, β, γ, ..., are the roots of the equation $\sin mx - nx \cos mx = 0$, prove that $\tan^{-1}\frac{x}{\alpha} + \tan^{-1}\frac{x}{\beta} + \ldots + \tan^{-1}\frac{x}{\nu} = 0$.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

This is problem 16, p. 328, Loney's *Trigonometry*. The problem is not clearly stated. In a letter December 2, 1908, from the author, he says he cannot now recall what he had in mind when he proposed the problem. As the problem reads, one would infer that if any ν roots of the given equation be taken and arcs formed whose tangents are any number, x, divided by these roots, the sum of these arcs is zero. But this is manifestly incorrect.

There is one way in which the statement is correct, viz: Suppose $\sin mx$ and $\cos mx$ be developed in series. The given equation then becomes

$$(m-n)x + m^2\left(\frac{n}{2\,!} - \frac{m}{3\,!}\right)x^3 - m^4\left(\frac{n}{4\,!} - \frac{m}{5\,!}\right)x^5 + \ldots \textit{ ad infinitum } = 0 \ldots (1).$$

One root of this equation is 0. Dividing the equation by x, we obtain

$$(m-n) + m^2\left(\frac{n}{2\,!} - \frac{m}{3\,!}\right)x^2 - m^4\left(\frac{n}{4\,!} - \frac{m}{5\,!}\right)x^4 + \ldots = 0 \ldots (2).$$

The roots of this equation, of which there are an infinite number, enter in pairs with opposite signs. Thus if $\alpha, \beta, \gamma, \ldots$ are roots, so are $-\alpha$, $-\beta$, $-\gamma$, ..., since the left hand member is an even function of x. Then, if the root, 0, is excluded, and if no arc exceeds, in absolute value, π radians, we have

$$\tan^{-1}\frac{x}{\alpha} + \tan^{-1}\frac{x}{-\alpha} + \tan^{-1}\frac{x}{\beta} + \tan^{-1}\frac{x}{-\beta} + \ldots + \tan^{-1}\frac{x}{\nu} + \tan^{-1}\frac{x}{-\nu}\Bigg]_{\nu=\infty} = 0.$$

The statement is also true if we take any $\nu/2$ pairs of roots.

If ν is increased indefinitely so that all roots except 0 are included, the statement, that the sum of the arcs$=n'\pi$, may be shown to be true, as follows: We have

$$\tan\left[\tan^{-1}\frac{x}{\alpha}+\tan^{-1}\frac{x}{\beta}+\tan^{-1}\frac{x}{\gamma}+\ldots+\tan^{-1}\frac{x}{\nu}\right].$$

$$=\frac{x\,\Sigma\frac{1}{\alpha}-x^3\,\Sigma\frac{1}{\alpha\beta\gamma}+x^5\,\Sigma\frac{1}{\alpha\beta\gamma\delta\varepsilon}-\ldots}{1-x^2\,\Sigma\frac{1}{\alpha\beta}+x^4\,\Sigma\frac{1}{\alpha\beta\gamma\delta}-\ldots}=0\ldots(3)\text{; for}$$

$$\Sigma\frac{1}{\alpha}\equiv\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\ldots=\frac{\beta\gamma\delta\varepsilon\ldots+\alpha\gamma\delta\epsilon\ldots+\alpha\beta\delta\epsilon\ldots+}{\alpha\beta\gamma\delta\varepsilon\ldots}.$$

The numerator of this fraction is the sum of the roots taken one less than all at a time, and is therefore equal to the coefficient of x which in equation (2) is zero, and the denominator is the product of the roots which in (2) is the known term, $m-n$. Hence, $\Sigma\frac{1}{\alpha}=0$. Similarly, $\Sigma\frac{1}{\alpha\beta\gamma}$ is a fraction whose numerator is the sum of the products of the roots taken three less than all at a time, and is therefore the coefficient of x^3 in (2), which is 0. Hence, $\Sigma\frac{1}{\alpha\beta\gamma}=0$. Similarly, for the other terms of the numerator of (3). From similar considerations, $\Sigma\frac{1}{\alpha\beta}$, $\Sigma\frac{1}{\alpha\beta\gamma\delta}$, $\ldots\neq0$.

Hence, $\tan\Sigma\left(\tan^{-1}\frac{x}{\alpha}\right)=0$. Hence, $\Sigma\tan^{-1}\frac{x}{\alpha}=n'\pi$ where n' is any integral positive or negative number.

Also solved by G. B. M. Zerr, and V. M. Spunar.

301. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

A is at Philadelphia, B at Chicago. A's personal equation is e; B's is E. When a star crosses A's meridian at time $t_1=8$ hours, 33 minutes, 24 seconds, he presses a button, telegraphing the fact to B, who receives it at time $t_2=7$ hours, 43 minutes, 23 seconds. When it crosses B's meridian at time $T_2=8$ hours, 33 minutes, 10 seconds, he telegraphs A, who receives it at time $T_1=9$ hours, 23 minutes, 11 seconds. They now exchange places, and on the second day following, B observes the transit at time $t'_1=8$ hours, 33 minutes, 26 seconds, and A gets the information at Chicago at time $t'_2=7$ hours, 43 minutes, 25 seconds. It crosses A's meridian at time $T'_2=8$ hours, 33 minutes, 12 seconds, and B gets the information at time $T'_1=9$ hours, 23 minutes, 13 seconds. Find the difference of longitude between Philadelphia and Chicago.

Solution by the PROPOSER.

The true times of the two transits at Philadelphia are t_1+e and T_1.

Hence, difference of time between Philadelphia and Chicago is $D=T_1+\ldots-t_1-e\ldots(1)$.